

Asymptotic bounds on renewal process stopping times

Jesse Geneson

geneson@gmail.com

Abstract

Suppose that i.i.d. random variables X_1, X_2, \dots are chosen uniformly from $[0, 1]$, and let $f : [0, 1] \rightarrow [0, 1]$ be an increasing bijection. Define μ_f to be the expected value of $f(X_i)$ for each i . Define the random variable K_f to be minimal so that $\sum_{i=1}^{K_f} f(X_i) > t$ and let $N_f(t)$ be the expected value of K_f . We prove that if $c_f = \frac{\int_0^1 \int_{f^{-1}(u)}^1 (f(x) - u) dx du}{\mu_f}$, then $N_f(t) = \frac{t + c_f}{\mu_f} + o(1)$. This generalizes a result of Čurgus and Jewett (2007) on the case $f(x) = x$.

1 Introduction

Renewal theory is a branch of mathematics with applications to waiting time distributions in queueing theory, ruin probabilities in insurance risk theory, the development of the age distribution of a population, and debugging programs [6, 8]. In this paper, we compare renewal processes, which are simple point processes $0 = x_0 < x_1 < x_2 < \dots$ for which the differences $x_{i+1} - x_i$ for each $i \geq 0$ form an independent identically distributed sequence.

A famous problem about renewal processes was actually a problem from the 1958 Putnam exam [1]: Select numbers randomly from the interval $[0, 1]$ until the sum is greater than 1. What is the expected number of selections?

The answer is e and solutions have appeared in several papers [1, 2, 3, 4, 5]. A more general problem is to find the expected number of selections until the sum is greater than t . Let $M(t)$ denote this expected number. In [7], Čurgus and Jewett showed that $M(t) = 2t + \frac{2}{3} + o(1)$ and $M(t) = \sum_{k=0}^{\lceil t \rceil} \frac{(-1)^k (t-k)^k}{k!} e^{t-k}$ [7].

An analogous question for products was posed in [10]: Select numbers randomly from the interval $[1, e]$ until the product is greater than e . What is the expected number of selections?

Vandervelde found that the answer is $\frac{e-1}{e} + e^{\frac{1}{e-1}}$ and posed the more general question of finding the number of selections until the product is greater than e^t [10]. Let $N(t)$ denote this expected number. Vandervelde conjectured that $N(t) \leq M(t)$ for all $t \geq 0$.

We prove the conjecture in Section 4, as well as the fact that $N(t) = (e-1)(t + \frac{e-2}{2}) + o(1)$. We use the same proof to obtain the following more general result.

Theorem 1. *Suppose that i.i.d. random variables X_1, X_2, \dots are chosen uniformly from $[0, 1]$, and let $f : [0, 1] \rightarrow [0, 1]$ be an increasing bijection. Define μ_f to be the expected value of $f(X_i)$ for each i . Define the random variable K_f to be minimal so that $\sum_{i=1}^{K_f} f(X_i) > t$ and let $N_f(t)$ be the expected value of K_f . If $c_f = \frac{\int_0^1 \int_{f^{-1}(u)}^{f^{-1}(u)+f(x)-u} dx du}{\mu_f}$, then $N_f(t) = \frac{t+c_f}{\mu_f} + o(1)$.*

As a corollary, this gives an alternative proof of the main result in [7], which was proved in that paper using results about delay functions.

Corollary 2. $M(t) = 2t + \frac{2}{3} + o(1)$

In Section 2, we find that $N(t) = \frac{e-1}{e} + e^{t-1+\frac{t}{e-1}}$ for $t \in [0, 1]$. We prove in Section 3 that $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}$, and we use this equation to find $N(t)$ for $t \in [1, 2]$.

2 $t \in [0, 1]$

The proof for $t \in [0, 1]$ is like the proof for $t = 1$ in [10].

Let $q_n = q_n(t)$ be the probability that a product of n numbers chosen from $[1, e]$ is not greater than e^t . Define $q_0 = 1$. The probability that the product exceeds e^t for the first time at the n^{th} selection is $(1 - q_n) - (1 - q_{n-1}) = q_{n-1} - q_n$. $N(t)$ is equal to $\sum_{n=1}^{\infty} n(q_{n-1} - q_n) = \sum_{n=0}^{\infty} q_n$.

For $t \in [0, 1]$ the region $R_n(t)$ within the n -cube $[1, e]^n$ consisting of points (x_1, \dots, x_n) , the product of whose coordinates is at most e^t , is described by $1 \leq x_1 \leq e^t, 1 \leq x_2 \leq \frac{e^t}{x_1}, \dots, 1 \leq x_n \leq \frac{e^t}{x_1 \dots x_{n-1}}$.

It is easy to see that $q_n = \frac{1}{(e-1)^n} \int_{R_n} dx_n \dots dx_1$, so we focus on computing $\Theta_n = \int_{R_n} dx_n \dots dx_1$. Note that $\Theta_{n+1} = \int_{R_n} (\frac{e^t}{x_1 \dots x_n} - 1) dx_n \dots dx_1$. Therefore $\Theta_{n+1} + \Theta_n = \int_{R_n} \frac{e^t}{x_1 \dots x_n} dx_n \dots dx_1$.

Lemma 3. $\Theta_n = (-1)^n(1 - b_n e^t)$, where $b_n = 1 - \frac{t}{1} + \frac{t^2}{2} - \dots + (-1)^{n-1} \frac{t^{n-1}}{(n-1)!}$

Proof. Make a change of variables $y_k = \ln x_k$, so $\Theta_{n+1} + \Theta_n = \int_{R'_n} e^t dy_n \dots dy_1$. Clearly R'_n consists of the points (y_1, \dots, y_n) satisfying $y_k \in [0, 1]$ and $y_1 + \dots + y_n \leq t$. Therefore $\Theta_{n+1} + \Theta_n = \frac{e^t t^n}{n!}$ for $t \in [0, 1]$.

For $n \geq 1$ let b_n be the n^{th} partial sum of the Taylor series for e^{-t} centered at 0, i.e., $1 - \frac{t}{1} + \frac{t^2}{2} - \dots + (-1)^{n-1} \frac{t^{n-1}}{(n-1)!}$. We show that $\Theta_n = (-1)^n(1 - b_n e^t)$. The quantities agree for $n = 1$. For $n \geq 2$, $\Theta_{n+1} + \Theta_n = (-1)^n(1 - b_n e^t) + (-1)^{n+1}(1 - b_{n+1} e^t) = e^t(-1)^n(b_{n+1} - b_n) = e^t(-1)^n \frac{(-1)^n t^n}{n!} = \frac{e^t t^n}{n!}$. ■

Therefore for $t \in [0, 1]$, $q_n = \frac{(-1)^n(1 - b_n e^t)}{(e-1)^n}$. The final step is to calculate the sum of the q_n .

Theorem 4. For $t \in [0, 1]$, $N(t) = \frac{e-1}{e} + e^{t-1+\frac{t}{e-1}}$

Proof. $\sum_{n=0}^{\infty} q_n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(e-1)^n} - \sum_{n=1}^{\infty} \frac{(-1)^n(b_n e^t)}{(e-1)^n} = \frac{e-1}{e} - \sum_{n=1}^{\infty} \frac{(-1)^n(b_n e^t)}{(e-1)^n}$.

We evaluate the remaining term by writing b_n as a sum and interchanging the order of summation.

$$\begin{aligned} - \sum_{n=1}^{\infty} \frac{(-1)^n(b_n e^t)}{(e-1)^n} &= -e^t \sum_{n=1}^{\infty} \frac{(-1)^n}{(e-1)^n} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} = \\ &= -e^t \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{n=k+1}^{\infty} \frac{(-1)^n}{(e-1)^n} = -e^t \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{(-1)^{k+1}}{e(e-1)^k} = \frac{e^t}{e} e^{\frac{t}{e-1}}. \end{aligned} \quad \blacksquare$$

3 $t \geq 1$

In this section, we show that $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}$ for $t \geq 1$ and calculate $N(t)$ for $t \in [1, 2]$.

Theorem 5. $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}$

Proof. In the next section we show that $N(t) = 1 + \frac{1}{e-1} \int_1^e N(t - \ln u) du$. If $s = t - \ln u$, then $N(t) = 1 + \frac{1}{e-1} e^t \int_{t-1}^t N(s) e^{-s} ds$. Therefore $N'(t) = \frac{e}{e-1}(N(t) - N(t-1)) - 1$, so $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}$. ■

Theorem 6. $N(t) = e^{\frac{e}{e-1}t}(-\frac{e-1}{e^{2+\frac{1}{e-1}}} + \frac{1}{e} + \frac{e^{-\frac{e}{e-1}}}{e-1}) + \frac{2(e-1)}{e} - \frac{e^{-\frac{e}{e-1}t} e^{\frac{e}{e-1}t}}{e-1}$ for $t \in [1, 2]$

Proof. By Theorems 4 and 5, $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -2e^{-\frac{e}{e-1}t} - \frac{e^{-\frac{e}{e-1}t}}{e-1}$. Therefore for $t \in [1, 2]$, $N(t)e^{-\frac{e}{e-1}t} = C + \frac{2(e-1)}{e}e^{-\frac{e}{e-1}t} - \frac{e^{-\frac{e}{e-1}t}}{e-1}$ for a constant $C = -\frac{e-1}{e^{2+\frac{1}{e-1}}} + \frac{1}{e} + \frac{e^{-\frac{e}{e-1}}}{e-1}$. In other words, $N(t) = e^{\frac{e}{e-1}t}(-\frac{e-1}{e^{2+\frac{1}{e-1}}} + \frac{1}{e} + \frac{e^{-\frac{e}{e-1}}}{e-1}) + \frac{2(e-1)}{e} - \frac{e^{-\frac{e}{e-1}t}}{e-1}$. ■

For each integer $i \geq 2$, $N(t)$ can be calculated similarly for $t \in [i, i+1]$ based on the values of $N(t)$ for $t \in [i-1, i]$ using the fact that $\frac{d}{dt}(N(t)e^{-\frac{e}{e-1}t}) = -\frac{e}{e-1}e^{-\frac{e}{e-1}t}N(t-1) - e^{-\frac{e}{e-1}t}$.

4 Bounds on $N(t)$

The results in this section use the fact that $\ln(1 + (e-1)t) \geq t$ for $t \in [0, 1]$.

Lemma 7. $\ln(1 + (e-1)t) \geq t$ for $t \in [0, 1]$

Proof. Define $f(t) = \ln(1 + (e-1)t) - t$ for $t \in [0, 1]$. Then $f'(t) = \frac{e-1}{1+(e-1)t} - 1$, so $f'(\frac{e-2}{e-1}) = 0$. Clearly $f'(t) > 0$ for $t \in [0, \frac{e-2}{e-1})$, $f'(t) < 0$ for $t \in (\frac{e-2}{e-1}, 1]$, and $f(0) = f(1) = 0$. Therefore $f(t) \geq 0$ for $t \in [0, 1]$. ■

The proof of the $N(t)$ recurrence is like the proof of the $M(t)$ recurrence in [7]. Let $I = [0, 1]$ and define $B_{0,t} = I^{\mathbb{N}}$. For each $n \in \mathbb{N}$, define $B_{n,t} = \{x \in I^{\mathbb{N}} : \ln(1 + (e-1)x_1) + \dots + \ln(1 + (e-1)x_n) \leq t\}$. Let $B = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} B_{n,k}$. Clearly the measure of B in $I^{\mathbb{N}}$ is 0, since $\ln(1 + (e-1)x_1) + \dots + \ln(1 + (e-1)x_n) \geq x_1 + \dots + x_n$.

Theorem 8. $N(t) = 1 + \frac{1}{e-1} \int_1^e N(t - \ln u) du$

Proof. Let $t \geq 0$ and define the random variable $F_t : I^{\mathbb{N}} \rightarrow \mathbb{N} \cup \{\infty\}$ by $F_t(x) = \min \{n \in \mathbb{N} : \ln(1 + (e-1)x_1) + \dots + \ln(1 + (e-1)x_n) > t\}$, with $\min \emptyset = \infty$. Since B has measure 0 and $F_t^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} B_{n,t} \subset B$, F_t is finite almost everywhere on $I^{\mathbb{N}}$.

For $n \in \mathbb{N}$, $F_t^{-1}(\{n\}) = B_{n-1,t} - B_{n,t}$. Thus F_t is a Borel function and $N(t) = \int_{I^{\mathbb{N}}} F_t(x) dx$. If $t \geq 1$ and $x = (x_1, x_2, x_3, \dots) = (w, v_1, v_2, \dots) = (w; v) \in I^{\mathbb{N}}$, then $2 \leq F_t(x) \leq \infty$ and $F_t(x) = F_t(w; v) = 1 + F_{t - \ln(1 + (e-1)w)}(v)$.

By Fubini's theorem, $N(t) = \int_{I^{\mathbb{N}}} F_t(x) dx = \int_0^1 \int_{I^{\mathbb{N}}} F_t(w; v) dv dw = \int_0^1 (1 + \int_{I^{\mathbb{N}}} F_{t - \ln(1 + (e-1)w)}(v) dv) dw = 1 + \int_0^1 N(t - \ln(1 + (e-1)w)) dw$. If $u = 1 + (e-1)w$, then $N(t) = 1 + \frac{1}{e-1} \int_1^e N(t - \ln u) du$. ■

Theorem 9. $M(t) \geq N(t)$ for all $t \geq 0$

Proof. As in the last proof, define $F_t : I^{\mathbb{N}} \rightarrow \mathbb{N} \cup \{\infty\}$ so that $F_t(x) = \min \{n \in \mathbb{N} : \ln(1 + (e-1)x_1) + \dots + \ln(1 + (e-1)x_n) > t\}$. Moreover, define $G_t : I^{\mathbb{N}} \rightarrow \mathbb{N} \cup \{\infty\}$ by $G_t(x) = \min \{n \in \mathbb{N} : x_1 + \dots + x_n > t\}$. Since $\ln(1 + (e-1)t) \geq t$ for $t \in [0, 1]$, then $F_t(x) \leq G_t(x)$ for all $t \geq 0$ and $x \in I^{\mathbb{N}}$. Thus $N(t) \leq M(t)$ for all $t \geq 0$. ■

We use Wald's equation to derive bounds on $N(t)$.

Theorem 10. (Wald's equation) Let X_1, X_2, \dots be i.i.d. random variables with common finite mean, and let τ be a stopping time which is independent of $X_{\tau+1}, X_{\tau+2}, \dots$ for which $E(\tau) < \infty$. Then $E(X_1 + \dots + X_\tau) = E(\tau)E(X_1)$.

Lemma 11. For all $t \geq 0$, $(e-1)t < N(t) \leq (e-1)(t+1)$.

Proof. Suppose that i.i.d. random variables X_1, X_2, \dots are chosen uniformly from $[0, 1]$. Define μ to be the expected value of $\ln(1 + (e-1)X_i)$ for each i . Define the random variable K to be minimal so that $\sum_{i=1}^K \ln(1 + (e-1)X_i) > t$ and define $S(t)$ to be the expected value of $\sum_{i=1}^K \ln(1 + (e-1)X_i)$. By definition, $N(t)$ is the expected value of K .

By Wald's equation, $N(t) = S(t)/\mu$, so $N(t) = (e-1)S(t)$. Since $t < S(t) \leq t+1$, then $(e-1)t < N(t) \leq (e-1)(t+1)$. ■

In order to prove that $N(t) = (e-1)(t + \frac{e-2}{2}) + o(1)$, we use two more well-known results.

Theorem 12. (Chernoff's bound) Suppose X_1, X_2, X_3, \dots are i.i.d. random variables such that $0 \leq X_i \leq 1$ for all i . Set $S_n = \sum_{i=1}^n X_i$ and $\mu = E(S_n)$. Then, for all $\delta > 0$, $Pr(|S_n - \mu| \geq \delta\mu) \leq 2e^{-\frac{\delta^2\mu}{2+\delta}}$.

Theorem 13. (Local Limit Theorem [9]) Let X_1, X_2, \dots be i.i.d. copies of a real-valued random variable X of mean μ and variance σ^2 with bounded density and a third moment. Set $S_n = \sum_{i=1}^n X_i$, let $f_n(y)$ be the probability density function of $\frac{S_n - n\mu}{\sqrt{n}}$, and let $\phi(y)$ be the probability density function of the Gaussian distribution $\mathcal{N}(0, \sigma^2)$. Then $\sup_{y \in \mathbb{R}} |f_n(y) - \phi(y)| = O(\frac{1}{\sqrt{n}})$.

Theorem 14. $N(t) = (e-1)(t + \frac{e-2}{2}) + o(1)$

Proof. As in the last proof, suppose that i.i.d. random variables X_1, X_2, \dots are chosen uniformly from $[0, 1]$. Define μ to be the expected value and σ^2 to be the variance of $\ln(1 + (e - 1)X_i)$ for each i . Define the random variable K to be minimal so that $\sum_{i=1}^K \ln(1 + (e - 1)X_i) > t$ and define $S(t)$ to be the expected value of $\sum_{i=1}^K \ln(1 + (e - 1)X_i)$. By definition, $N(t)$ is the expected value of K .

By Wald's equation, $N(t) = S(t)/\mu$, so $N(t) = (e - 1)S(t)$. It remains to prove that $S(t) - t = \frac{e-2}{2} + o_t(1)$.

For each integer $i \geq 0$, define the random variable $Y_i = \sum_{j=1}^i \ln(1 + (e - 1)X_j)$ and let $p_i(u)$ be the probability density function for the random variable $U = (t - Y_i | Y_{i+1} > t \wedge Y_i \leq t)$. We will show that $p_i(u) = e - e^u + o_t(1)$ for $i \in [(e - 1)t - c\sqrt{t}, (e - 1)t + c\sqrt{t}]$ for all constants $c \geq 0$.

Define $q_i(y)$ to be the density function for Y_i . By Bayes' Theorem and the fact that $Pr(\ln(1 + (e - 1)X_i) \geq u) = 1 - \frac{e^u - 1}{e - 1}$, $p_i(u) = \frac{q_i(t - u)(1 - \frac{e^u - 1}{e - 1})}{\int_0^1 q_i(t - y)(1 - \frac{e^y - 1}{e - 1})dy}$.

Let $\phi(x)$ be the density function of the distribution $\mathcal{N}(0, \sigma^2)$. By the local limit theorem, $p_i(u) = \frac{(\frac{1}{\sqrt{i}}\phi(\frac{t - u - \frac{i}{e-1}}{\sqrt{i}}) \pm O(\frac{1}{i}))(1 - \frac{e^u - 1}{e - 1})}{\int_0^1 (\frac{1}{\sqrt{i}}\phi(\frac{t - y - \frac{i}{e-1}}{\sqrt{i}}) \pm O(\frac{1}{i}))(1 - \frac{e^y - 1}{e - 1})dy} = (e - 1)(1 - \frac{e^u - 1}{e - 1}) + o_t(1) = e - e^u + o_t(1)$ for $i \in [(e - 1)t - c\sqrt{t}, (e - 1)t + c\sqrt{t}]$.

Now define the random variable $O = -t + \sum_{i=1}^K \ln(1 + (e - 1)X_i)$, and let $O_i(t)$ be the expected value of $(O | K = i + 1)$. Furthermore define $V_{i,u}(t)$ to be the expected value of $(O | (U = u \wedge K = i + 1))$. Then for $i \in [(e - 1)t - c\sqrt{t}, (e - 1)t + c\sqrt{t}]$, $O_i(t) = \int_0^1 p_i(u)V_{i,u}(t)du = \int_0^1 (e - e^u + o_t(1))(\frac{1}{1 - \frac{e^u - 1}{e - 1}} \int_{\frac{e^u - 1}{e - 1}}^1 (\ln(1 + (e - 1)x) - u)dx)du = \frac{e-2}{2} + o_t(1)$.

For any $\epsilon > 0$, there is a constant $c = c(\epsilon) > 0$ such that $Pr(|K - 1 - (e - 1)t| > c\sqrt{t}) < \epsilon$ by Chernoff's bound. Therefore, there is a sequence $\epsilon_0 > \epsilon_1 > \epsilon_2 > \dots$ converging to 0 such that $|S(t) - t - \sum_{i=\lfloor (e-1)t - c(\epsilon_j)\sqrt{t} \rfloor}^{\lceil (e-1)t + c(\epsilon_j)\sqrt{t} \rceil} O_i(t)Pr(K = i + 1)| < \epsilon_j$. Thus $S(t) - t = \frac{e-2}{2} + o_t(1)$. ■

The proof above also generalizes to other functions f besides $f(x) = \ln(1 + (e - 1)x)$. In particular, $\ln(1 + (e - 1)x)$ can be replaced in the proof with an increasing bijection $f : [0, 1] \rightarrow [0, 1]$, thus implying Theorem 1.

Proof. Suppose that i.i.d. random variables X_1, X_2, \dots are chosen uniformly from $[0, 1]$. Define μ_f to be the expected value and σ_f^2 to be the variance of $f(X_i)$ for each i . Define the random variable K_f to be minimal so that

$\sum_{i=1}^{K_f} f(X_i) > t$ and define $S_f(t)$ to be the expected value of $\sum_{i=1}^{K_f} f(X_i)$. By definition, $N_f(t)$ is the expected value of K_f .

By Wald's equation, $N_f(t) = S_f(t)/\mu_f$. It remains to prove that $S_f(t) - t = \frac{\int_0^1 \int_{f^{-1}(u)}^1 (f(x)-u) dx du}{\mu_f} + o_t(1)$.

For each integer $i \geq 0$, define the random variable $Y_i = \sum_{j=1}^i f(X_j)$ and let $p_i(u)$ be the probability density function for the random variable $U = (t - Y_i | Y_{i+1} > t \wedge Y_i \leq t)$. We will show that $p_i(u) = \frac{1-f^{-1}(u)}{\mu_f} + o_t(1)$ for $i \in [\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}]$ for all constants $c \geq 0$.

Define $q_i(y)$ to be the density function for Y_i . By Bayes' Theorem and the fact that $Pr(f(X_i) \geq u) = 1 - f^{-1}(u)$, $p_i(u) = \frac{q_i(t-u)(1-f^{-1}(u))}{\int_0^1 q_i(t-y)(1-f^{-1}(y)) dy}$.

Let $\phi(x)$ be the density function of the distribution $\mathcal{N}(0, \sigma_f^2)$. By the local limit theorem, $p_i(u) = \frac{(\frac{1}{\sqrt{i}} \phi(\frac{t-u-i\mu_f}{\sqrt{i}}) \pm O(\frac{1}{i}))(1-f^{-1}(u))}{\int_0^1 (\frac{1}{\sqrt{i}} \phi(\frac{t-y-i\mu_f}{\sqrt{i}}) \pm O(\frac{1}{i}))(1-f^{-1}(y)) dy} = \frac{(1-f^{-1}(u))}{\int_0^1 (1-f^{-1}(y)) dy} + o_t(1) = \frac{(1-f^{-1}(u))}{\mu_f} + o_t(1)$ for $i \in [\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}]$.

Now define the random variable $O = -t + \sum_{i=1}^{K_f} f(X_i)$, and let $O_i(t)$ be the expected value of $(O | K_f = i + 1)$. Furthermore define $V_{i,u}(t)$ to be the expected value of $(O | (U = u \wedge K_f = i + 1))$. Then for $i \in [\frac{t}{\mu_f} - c\sqrt{t}, \frac{t}{\mu_f} + c\sqrt{t}]$, $O_i(t) = \int_0^1 p_i(u) V_{i,u}(t) du = \int_0^1 (\frac{(1-f^{-1}(u))}{\mu_f} + o_t(1)) (\frac{1}{1-f^{-1}(u)} \int_{f^{-1}(u)}^1 (f(x) - u) dx) du = \frac{\int_0^1 \int_{f^{-1}(u)}^1 (f(x) - u) dx du}{\mu_f} + o_t(1)$.

For any $\epsilon > 0$, there is a constant $c = c(\epsilon) > 0$ such that $Pr(|K_f - 1 - \frac{t}{\mu_f}| > c\sqrt{t}) < \epsilon$ by Chernoff's bound. Therefore, there is a sequence $\epsilon_0 > \epsilon_1 > \epsilon_2 > \dots$ converging to 0 such that $|S_f(t) - t - \sum_{i=\lfloor \frac{t}{\mu_f} - c(\epsilon_j)\sqrt{t} \rfloor}^{\lceil \frac{t}{\mu_f} + c(\epsilon_j)\sqrt{t} \rceil} O_i(t) Pr(K_f = i + 1)| < \epsilon_j$. Thus $S_f(t) - t = \frac{\int_0^1 \int_{f^{-1}(u)}^1 (f(x) - u) dx du}{\mu_f} + o_t(1)$. ■

References

- [1] L. Bush. The William Putnam mathematical competition. Amer Math Monthly 68 (1961) 18-33.
- [2] H. Shultz. An expected value problem. Two-Year College Math J 10 (1979) 179.

- [3] N. MacKinnon. Another surprising appearance of e . Math Gazete 74 (1990) 167-9.
- [4] S. Schwartzman. An unexpected expected value. Math Teacher (1993) 118-20.
- [5] E. Weisstein. Uniform sum distribution. From MathWorld. <http://mathworld.wolfram.com/UniformSumDistribution.html>.
- [6] J. Blanchet and P. Glynn. Uniform Renewal Theory with Applications to Expansions of Random Geometric Sums. Advances in Applied Probability 39(4) (2007) 1070-1097.
- [7] B. Ćurgus and R. I. Jewett, An unexpected limit of expected values, Expo. Math. 25 (2007), 1-20.
- [8] J. Doob. Renewal Theory from the Point of View of the Theory of Probability. Transactions of the AMS 63(3) (1947) 422-438.
- [9] L. Hervé and J. Ledoux. Local limit theorem for densities of the additive component of a finite Markov Additive Process. Statistics and Probability Letters 83 (2013) 2119-2128.
- [10] S. Vandervelde. Expected Value Road Trip, Mathematical Intelligencer, 30(2) (2008) 17-18.